International Journal of Engineering, Science and Mathematics (UGC Approved)
Vol. 7 Issue 3, March 2018
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gate as well as in Cabell's Directories of Publishing Opportunities, U.S.A

ON THE LAPLACE TRANSFORMS OF THE GENERALIZED HYPERGEOMETRIC FUNCTIONS ${}_pF_p$ - I

Lalit Mohan Upadhyaya [*]

2010 AMS Mathematics Subject Classification:
Primary: 33C20, 44A10, 44A45.
Secondary: 33C05, 33C90.

Keywords:

Generalized hypergeometric functions ${}_pF_p$, Laplace transform, summation theorems.

Abstract

In this paper we intend to point out some minor typos which might have crept in inadvertently in four of the very recent results deduced by Kim and Lee [20] concerning the Laplace transforms of certain generalized hypergeometric functions ${}_pF_p$. We also augment their study [20] by presenting three additional results for which Kim and Lee [20] have stated the governing preliminary results in their introductory part of this paper [20] but the corresponding results flowing from these preliminary results have neither been stated nor deduced by them in the sequel. In this sense this study of ours augments the above mentioned investigations of Kim and Lee [20].

Author correspondence:

Lalit Mohan Upadhyaya,
Department of Mathematics,
Municipal Post Graduate College, Mussoorie,
Dehradun, Uttarakhand, India -248179
Email: lmupadhyaya@rediffmail.com

1. Introduction

The subject of operational calculus is an important branch of mathematics which opens many vistas for the application of mathematics to solve either analytically or numerically many problems which arise in physics, applied mathematics, engineering and other relevant disciplines of study including astronomy, etc. We as students and teachers of mathematics at the advanced undergraduate or post graduate level and also as students of engineering and physics courses are most frequently confronted first of all with the introduction of Laplace transforms in our courses in operational calculus. The domain of application of the Laplace transforms and other integral transforms in applied

[*] Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand, India - 248179. Email: lmupadhyaya@rediffmail.com

mathematics, engineering and physics problems is so vast that mathematicians over the decades have come up with various extensions of the classical Laplace transform – like the Laplace transform for functions of matrix arguments [9, 10], the q-Laplace transform (i.e., the basic analogue of the Laplace transform) [11, 17, 18, 19], the degenerate Laplace transform [12-16], etc. The ever growing applications of the Laplace transforms in different and diverse branches of study have led many researchers to explore the Laplace transforms of many special functions and record them in mathematical literature [1-6] so that these results may be found useful by the research community in other disciplines to solve the problems encountered by them in their respective research fields where these results have often come to their rescue as handy mathematical tools to crack the complex problems in hand.

A number of researchers have devoted their painstaking efforts to find the Laplace transforms of a number of special functions and these are very widely scattered in the literature. Systematic efforts have been made to collect these formulae and record them in a consolidated manner in the form of authoritative handbooks [1-6]. The Laplace transforms of many widely used special functions can be found tabulated in [2-6]. Mathai [10] discusses the Laplace transforms of a number of multiple hypergeometric functions of matrix arguments with real symmetric positive definite matrices and Hermitian positive definite matrices as arguments. The author has also generalized a number of Laplace type integrals of multiple hyprgeometric functions of matrix arguments in [9] and in a number of his preprints which can be found in the references listed in [9]. In a very recent paper Kim and Lee [20] have established the Laplace transforms of some generalized hypergeometric functions of the type ${}_pF_p$. While going through this paper the author found some minor typing errors in four out of the seven results established in that paper. Further, the authors of [20] have also stated some preliminary results in the introductory part of their paper, but they have not given the Laplace transforms of the corresponding generalized hypergeometric functions in the second part of their paper which can be deduced by using these results. Therefore, in this paper we aim to point out the minor typos in the four results of Kim and Lee [20] and also deduce the Laplace transforms of three remaining hypergeometric functions of the type ${}_pF_p$ for which the preliminary results are already stated in the first section of [20].

The paper consists of two sections – the necessary background material and essential results are stated in the first section of the paper, while the three main results are deduced in the second section of the paper with the help of these foundation results.

It is well known that the Laplace transform of a function $f(t)$ of the variable t defined for $t>0$, denoted by $\mathbf{L}\{f(t)\}$, is defined by the integral (see, for instance, [8, (1), p.1], or, Chapter 3 [21])

$$\mathbf{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{1.1}$$

if the integral in (1.1) converges for some value of the complex parameter s. For sufficient conditions for the existence of the Laplace transform of a function $f(t)$, we refer the reader to the Theorem 1-1, p.2 and the Problem 145, p.38 of [8] or, to the Section 3.3 of [21]. The generalized multiple hypergeometric function, denoted by ${}_pF_q$, having p numerator parameters $\alpha_1,\cdots,\alpha_p$ and q denominator parameters $\beta_1,\cdots,\beta_q$ is defined by the following equation (see, for instance, (23), p.19 [22])

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};z\right] = {}_pF_q\left[\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \tag{1.2}$$

where, $(a)_n = a(a+1)\cdots(a+n-1);(a)_0 = 1$ is the Pochhammer symbol (shifted factorial). In the multiple hypergeometric series (1.2), p and q are positive integers or zero and the numerator parameters $\alpha_1,\cdots,\alpha_p$ and the denominator parameters $\beta_1,\cdots,\beta_q$ are complex numbers and

$$\beta_j \neq 0,-1,-2,\ldots;\ j=1,2,\ldots,q \tag{1.3}$$

Then, if none of the numerator parameters in (1.2) is zero or negative (see, p.20 [22]), the series in (1.2) under the restrictions (1.3) – (i) converges for $|z|<\infty$ if $p\le q$; (ii) converges for $|z|<1$ if $p=q+1$ and (iii) diverges for all $z, z\neq 0$, if $p>q+1$. Again, if we let $\omega=\sum_{i=1}^{q}\beta_i-\sum_{j=1}^{p}\alpha_j$ then the series ${}_pF_q$, with $p=q+1$ is – (i) absolutely convergent for $|z|=1$ if $\mathrm{Re}(\omega)>0$; (ii) conditionally convergent for $|z|=1, z\neq 1$ if $-1<\mathrm{Re}(\omega)\le 0$, and (iii) divergent for $|z|=1$ if $\mathrm{Re}(\omega)\le -1$.

We mention the following result from [6] (see 3.38.1.1, p. 546 [6])

$$\int_0^{\infty} e^{-px}x^{\mu}\,{}_mF_n\left[(a_m);(b_n);-\omega x\right]dx = \frac{\Gamma(\mu+1)}{p^{\mu+1}}\,{}_{m+1}F_n\left[\begin{matrix}\mu+1,(a_m)\\ (b_n)\end{matrix};\frac{-\omega}{p}\right] \tag{1.4}$$

where $(a_m)=a_1,\cdots,a_m;(b_n)=b_1,\cdots,b_n$ $[m+n\neq 0; a_j\neq 0,-1,-2,\ldots;\ j=1,2,\ldots,m;$ any of the the following three groups of conditions holds: 1) $m=n>0;\mathrm{Re}\,\mu>-1;\mathrm{Re}\,\omega\ge 0;\mathrm{Re}\,p>0;$ 2) $m=n>0;\ \mathrm{Re}\,\mu>-1;\mathrm{Re}\,p>0;\pi/2<|\arg\omega|<3\pi/2$; 3) $m\le n-1;\mathrm{Re}\,\mu>-1;\mathrm{Re}\,p>0;$ ω is arbitrary]. Letting $p=s,\mu=\nu-1,m=p,n=q,\omega\to-\omega,a=\alpha,b=\beta,x=t$ changes (1.4) into

$$\int_0^{\infty} e^{-st}t^{\nu-1}\,{}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};\omega t\right]dt = \Gamma(\nu)s^{-\nu}\,{}_{p+1}F_q\left[\begin{matrix}\nu,\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};\frac{\omega}{s}\right] \tag{1.5}$$

with one of the following conditions holding: 1) when $p<q,\mathrm{Re}\,\nu>0,\mathrm{Re}\,s>0$ and ω is arbitrary; 2) when $p=q>0,\mathrm{Re}\,\nu>0,\mathrm{Re}\,s>\mathrm{Re}\,\omega$. This result is also stated in Kim and Lee [20] as (2) on p.74[20].

We now list the fundamental results on the basis of which the results missing in the paper of Kim and Lee [20] will be established in section 2 of this paper or the corrections (minor typos) will be pointed out in their established results. The following result is given in (18), p.77 of Kim and Lee [20]. For a nonnegative integer n,

$${}_3F_2\left[\begin{matrix}-n, & a, & c+1\\ & b, & c\end{matrix};1\right] = \frac{(\alpha+n)}{\alpha}\frac{(b-a-1)_n}{(b)_n},\alpha=\frac{c(1+a-b)}{a-c} \tag{1.6}$$

The following summation theorems can be found in Slater [7] at the indicated positions. Kim and Lee [20] also mention these results in their paper at the places as noted below:

$$ {}_4F_3\left[\begin{matrix} a,\ 1+\frac{a}{2},\quad b,\ c \\ \frac{a}{2}, 1+a-b, 1+a-c \end{matrix};1\right] = \frac{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-b-c\right)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}-b\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-c\right)\Gamma(1+a)\Gamma(1+a-b-c)} \tag{1.7} $$

Slater ([7] (III.22), p.245); Kim and Lee ([20], (20), p.77).

$$ {}_5F_4\left[\begin{matrix} a,\ 1+\frac{a}{2},\quad b,\quad c,\quad d \\ \frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix};1\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \tag{1.8} $$

Slater ([7], (III.12), p.244); Kim and Lee ([20], (25), p.78).

$$ {}_6F_5\left[\begin{matrix} a,\ 1+\frac{a}{2},\quad b,\quad c,\quad d,\quad e \\ \frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e \end{matrix};-1\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(a)\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(a+c+e)}\times $$

$$ \frac{\Gamma\left(1+\frac{1}{2}+\frac{a}{2}-\frac{b}{2}-\frac{c}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-\frac{d}{2}-\frac{e}{2}\right)}{\Gamma\left(1+\frac{a}{2}-\frac{b}{2}-\frac{d}{2}\right)\Gamma\left(1+\frac{a}{2}-\frac{c}{2}-\frac{e}{2}\right)} \tag{1.9} $$

where $1=b+c=d+e$. Slater ([7], (III.27), p.245); Kim and Lee ([20], (26), p.78).

$$ {}_4F_3\left[\begin{matrix} a,\ 1+\frac{a}{2},\quad b,\ -n \\ \frac{a}{2},\ 1+a-b,\ 1+2b-n \end{matrix};1\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n} \tag{1.10} $$

Slater ([7], (III.17), p.24); Kim and Lee([20], (22), p.77).

$$ {}_4F_3\left[\begin{matrix} a,\ 1+\frac{a}{2},\quad b,\ -n \\ \frac{a}{2},\ 1+a-b,\ 2+2b-n \end{matrix};1\right] = \frac{(a-2b-1)_n\left(\frac{a}{2}+\frac{1}{2}-b\right)_n(-b-1)_n}{(1+a-b)_n\left(\frac{a}{2}-\frac{1}{2}-b\right)_n(-2b-1)_n} \tag{1.11} $$

Slater ([7], (III.18), p.244), Kim and Lee ([20], (23), p.77).

$$ {}_4F_3\left[\begin{matrix} \frac{a}{2},\ \frac{1}{2}+\frac{a}{2},\quad b+n,\ -n \\ \frac{b}{2},\ \frac{1}{2}+\frac{b}{2},\ 1+a \end{matrix};1\right] = \frac{(b-a)_n}{(b)_n} \tag{1.12} $$

Slater ([7], (III.20), p.245); Kim and Lee ([20], (24), p.77).

2. Results for The Laplace Transforms of ${}_pF_p$

With the above results we now proceed first to restate some results (in their correct forms) concerning the Laplace transforms of certain hypergeometric functions of the type ${}_pF_p$ for particular values of p, which have been derived by Kim and Lee [20], but some minor typographical errors have crept in their published results, which (as far as this author thinks) may perhaps be due to some inadvertent errors in the typing of their manuscript of this paper [20]. For $p=q=2$, (1.5) yields ,as Kim and Lee [20] have noted in (10), p.75 of [20],

$$\int_0^\infty e^{-st}t^{\nu-1}{}_2F_2\left[\begin{matrix}a,b\\c,d\end{matrix};\omega t\right]dt=\Gamma(\nu)s^{-\nu}{}_3F_2\left[\begin{matrix}\nu,a,b\\c,d\end{matrix};\frac{\omega}{s}\right] \tag{2.1}$$

where $\operatorname{Re}\nu>0$ and $\operatorname{Re}s>\max(\operatorname{Re}\omega,0)$. In this relation if we put $\omega=s,\nu=a,a=-n,$ $b=c+1,d=b$ where, n is a non-negative integer, it yields, as recorded by Kim and Lee [20] in (29), p.79,

$$\mathbf{L}\left\{t^{a-1}{}_2F_2\left[\begin{matrix}-n,&c+1\\b,&c\end{matrix};st\right];s\right\}=\int_0^\infty e^{-st}t^{a-1}{}_2F_2\left[\begin{matrix}-n,c+1\\b,c\end{matrix};st\right]dt=\Gamma(a)s^{-a}{}_3F_2\left[\begin{matrix}-n,a,c+1\\b,c\end{matrix};1\right]$$

which yields,
$$\mathbf{L}\left\{t^{a-1}{}_2F_2\left[\begin{matrix}-n,&c+1\\b,&c\end{matrix};st\right];s\right\}=\frac{(\alpha+n)\Gamma(a)(b-a-1)_n}{s^a\alpha(b)_n} \tag{2.2}$$

with the help of (1.6). (2.2) gives the correct form of (29), p.79 of Kim and Lee [20], where α is given in (1.6).

Similarly, the correct form of (34), p.80 of Kim and Lee [20] as given with the aid of (1.7) is

$$\mathbf{L}\left\{t^{\frac{a}{2}}{}_3F_3\left[\begin{matrix}a,\ b,\ c\\\frac{a}{2},\ 1+a-b,\ 1+a-c\end{matrix};st\right];s\right\}$$
$$=\sqrt{\pi}s^{-\left(1+\frac{a}{2}\right)}\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(\frac{1}{2}+\frac{a}{2}-b-c\right)}{2^a\Gamma\left(\frac{1}{2}+\frac{a}{2}-b\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-c\right)\Gamma(1+a-b-c)} \tag{2.3}$$

We may mention here that the factor $\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)$ appearing in the numerator on the right side of (34), p.80 of Kim and Lee [20] should not appear there.

The correct form of (36), p.81 of Kim and Lee[20], given with the aid of (1.8) is

$$\mathbf{L}\left\{t^{b-1}{}_4F_4\left[\begin{matrix}a,\ 1+\frac{a}{2},\ c,\ d\\\frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix};st\right];s\right\}$$
$$=\Gamma(b)s^{-b}\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \tag{2.4}$$

It is worth mentioning here that the factor s^{b-1} appearing in the numerator on the right side of (36), p.81 of Kim and Lee[20] should, in fact be s^{-b}, their remaining expression on the right side of (36), p.81 is otherwise correct.

The correct form of (37), p.81 of Kim and Lee [20] can be had with the help of (1.9) as

$$\mathbf{L}\left\{t^{b-1}{}_5F_5\left[\begin{matrix}a,\ 1+\frac{a}{2},\ c,\ d,\ e\\\frac{a}{2},\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix};-st\right];s\right\}$$

$$=\Gamma(b)s^{-b}\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(a)\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(a+c+e)}\times$$

$$\frac{\Gamma\left(1+\frac{1}{2}+\frac{a}{2}-\frac{b}{2}-\frac{c}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}-\frac{d}{2}-\frac{e}{2}\right)}{\Gamma\left(1+\frac{a}{2}-\frac{b}{2}-\frac{d}{2}\right)\Gamma\left(1+\frac{a}{2}-\frac{c}{2}-\frac{e}{2}\right)} \tag{2.5}$$

where $1=b+c=d+e$.

We mention here that though Kim and Lee [20] have given three more summation formulas in (22), (23) and (24) on p.77 of their paper [20], which are precisely (1.10), (1.11) and (1.12) written above, and though they have explicitly declared in the lines (16) - (19) on p.76 of [20], just above (17), that "Motivated by these work, in our a present investigation, we aim to obtain new laplace transforms of generalized hypergeometric functions ${}_2F_2(x)$, ${}_3F_3(x)$, ${}_4F_4(x)$ and ${}_5F_5(x)$. For this, we shall need the following summation theorems ... given respectively by ...", yet, this author has found that they have nowhere used any of these three formulas in deducing any of their results for the Laplace transforms of the corresponding generalized hypergeometric functions in the second section of their paper [20]. Therefore, in order to complete this paper [20] of Kim and Lee, or say, to augment it, this author decided to write this paper. Below we give three additional formulas for the Laplace transforms of the generalized hypergeometric function ${}_3F_3$ which follow from the use of the formulas (22), (23) and (24) on p.77 of the paper of Kim and Lee [20] (i.e. (1.10), (1.11) and (1.12) above) with a view to augment and complete this paper of Kim and Lee [20].

If we put $p=q=3$ in (1.5) with the choice of the parameters $\alpha_1=a,\alpha_2=1+\frac{a}{2},\alpha_3=-n,\beta_1=\frac{a}{2},\beta_2=1+a-b,\beta_3=1+2b-n,\nu=b$ and $\omega=s$ where, n is a nonnegative integer and then use (1.10) (i.e. the formula (22) p.77 of Kim and Lee [20]) for the resulting ${}_4F_3$ function on the right side of (1.5) yields

$$\mathbf{L}\left\{t^{b-1}{}_3F_3\left[\begin{matrix} a,\ 1+\frac{a}{2},\ -n \\ \frac{a}{2},\ 1+a-b,\ 1+2b-n\end{matrix};st\right];s\right\}$$

$$=\Gamma(b)s^{-b}\frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n} \tag{2.6}$$

For the same choice of parameters in (1.5) as used in deducing (2.6) except, $\beta_3=2+2b-n$ followed by the application of (1.11) (i.e. the formula (23) p.77 of Kim and Lee[20]) to the ensuing ${}_4F_3$ function on the right side of (1.5) generates

$$\mathbf{L}\left\{t^{b-1}{}_3F_3\left[\begin{matrix} a,\ 1+\frac{a}{2},\ -n \\ \frac{a}{2},\ 1+a-b,\ 2+2b-n\end{matrix};st\right];s\right\}$$

$$=\Gamma(b)s^{-b}\frac{(a-2b-1)_n\left(\frac{a}{2}+\frac{1}{2}-b\right)_n(-b-1)_n}{(1+a-b)_n\left(\frac{a}{2}-\frac{1}{2}-b\right)_n(-2b-1)_n} \tag{2.7}$$

Finally, with $p=q=3$ and for the choice of parameters $\alpha_1=\frac{1}{2}+\frac{a}{2},\alpha_2=b+n,\alpha_3=-n,\beta_1=\frac{b}{2},\beta_2=\frac{b}{2}+\frac{1}{2},\beta_3=1+a,\nu=\frac{a}{2}$ and $\omega=s$ in (1.5) and then appealing to (1.12) (i.e. the formula (24) p.77 of Kim and Lee[20]) for the consequent ${}_4F_3$ function on the right side of (1.5) produces

$$L\left\{t^{\frac{a}{2}-1}{}_3F_3\left[\begin{matrix}\frac{1}{2}+\frac{a}{2}, & b+n, & -n\\ \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}, & 1+a\end{matrix};st\right];s\right\}$$

$$=\Gamma\left(\frac{a}{2}\right)s^{-\frac{a}{2}}\frac{(b-a)_n}{(b)_n} \tag{2.8}$$

We close the paper with the remark that we shall extend this work of Kim and Lee [20] in our forthcoming papers of this series.

Acknowledgements

The author would like to express his most sincere gratitude and thanks to The Hon’ble Professor (Dr.) Edward G. Dunne, Executive Editor, Mathematical Reviews/MathSciNet, American Mathematical Society, U.S.A., All The Hon’ble Editorial Board Members and All The Hon’ble Staff Members of the Mathematical Reviews/MathSciNet, American Mathematical Society, U.S.A. and The Hon’ble American Mathematical Society, U.S.A. for providing him the reference number [20] and also for providing him the most coveted opportunity to write a Review (number MR 3676675) for this paper [20] (which, however, could not be written by this author in time owing to some practical problems on his part), without which this author would not have been able to write this series of papers. The author would also like to express his most sincere gratitude and thanks to The Hon’ble Professors - Professor (Dr.) Klaus Hulek (Editor-in-Chief) and Professor (Dr.) Dirk Werner (Deputy Editor-in-Chief), zbMATH (formerly, Zentralblatt MATH), of the Hon’ble FIZ Karlsruhe – Leibniz Institute for Information Infrastructure Mathematics Department, zbMATH Franklinstr. 11, D-10587 Berlin, Germany, All The Hon’ble Editorial Board Members and All The Hon’ble Staff Members of the zbMATH Germany for providing him the reference number [21] and also for providing him the most coveted opportunity to write a Review (number Zbl 06840501) for this book [21]. The author, hereby, explicitly thanks and expresses his heartfelt gratitude towards both – the Mathematical Reviews/MathSciNet and the zbMATH – both of which are the World’s topmost and the most prestigious resources of mathematical research – for providing him their immense and invaluable help which is most positively influencing his most humble research work and he hereby acknowledges most gratefully the greatest and the most generous supports and the extremely helpful services that he is receiving from both of these two greatest institutions and leaders of Mathematical Research in the World.

References

[1] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. (1953). Higher Transcendental Functions, Vols. I, II, III McGraw Hill Book Company, New York, Toronto and London.

[2] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. (1954). Tables of Integral Transforms, Vols.I, II, McGraw Hill Book Company, New York, Toronto and London.

[3] F. Oberhettinger and L. Badii.(1973). Tables of Laplace Transforms, Springer-Verlag, New York, Heidelberg.

[4] Spiegel M.R. (1968). Schaum's Handbook of Mathematical Formulas, McGraw Hill , New York.

[5] Gradshteyn I.S., Ryzhik I.M. (2007). Tables of Integrals, Series and Products, Seventh Edition, (Edited by Alan Jeffrey and Daniel Zwillinger), Academic Press, California, U.S.A.

[6] Prudnikov A.P., Brychkov Yu.A. and Marichev O.I. (1992). Integrals and Series, Vol. 4, Direct Laplace Transforms, Gordon and Breach Science Publishers, New York, Paris, Melbourne, Tokyo.

[7] Slater L. J. (1966). Generalized Hypergeometric Functions, Cambridge University Press, Cambridge.

[8] Spiegel M.R. (1965). Laplace Transforms (Schaum's Outlines), McGraw Hill, New York.

[9] Upadhyaya Lalit Mohan (Nov. 2003): Matrix Generalizations of Multiple Hypergeometric Functions By Using Mathai's Matrix Transform Techniques (Ph.D. Thesis, Kumaun University, Nainital, Uttarakhand, India) #1943, *IMA Preprint Series*, University of Minnesota, Minneapolis, U.S.A.
(https://www.ima.umn.edu/sites/default/files/1943.pdf
http://www.ima.umn.edu/preprints/abstracts/1943ab.pdf
http://www.ima.umn.edu/preprints/nov2003/1943.pdf
http://hdl.handle.net/11299/3955
https://zbmath.org/?q=an:1254.33008
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.192.2172\&rank=52).

[10] Mathai A.M. (1997). Jacobians of Matrix Transformations and Functions of Matrix Argument World Scientific Publishing Co. Pte. Ltd., Singapore.

[11] Chung W. S., Kim T. and Kwon H. I. (2014). On the *q*-Analog of the Laplace Transform, *Russ. J. Math. Phys,* . **21** (2), 156–168.

[12] Kim T., Kim D.S. (2017). Degenerate Laplace Transform and Degenerate Gamma Function, *Russ. J. Math. Phys.,* 24(2), 241-248. MR3658414.

[13] Upadhyaya Lalit Mohan (2017-2018). On The Degenerate Laplace Transform – I, Communicated for possible publication to *Bulletin of Pure and Applied Sciences, Section E- Mathematics & Statistics*, Vol. 37, No. 1, (January – June) 2018.

[14] Upadhyaya Lalit Mohan (2017). On The Degenerate Laplace Transform – II, *International Journal of Engineering & Scientific Research*, Vol. 5, Issue 12, December 2017, 63-71. (http://esrjournal.com/uploads/91/4403_pdf.pdf)

[15] Upadhyaya Lalit Mohan (2018). On The Degenerate Laplace Transform – III, *International Journal of Engineering, Science and Mathematics*, Vol. 7, Issue 1, January 2018, 400-410.
(http://ijesm.co.in/abstract.php?article_id=4613&title=ON%20THE%20DEGENERATE%20LAPLACE%20TRANSFORM%20%20III)

[16] Upadhyaya Lalit Mohan (2018). On The Degenerate Laplace Transform – IV, *International Journal of Engineering & Scientific Research*, Vol. 6, Issue 2, February 2018, 198-209. (http://esrjournal.com/uploads/91/4863_pdf.pdf)

[17] Uçar Faruk and Albayrak Durmuş (2012). On q-Laplace Type Integral Operators and Their Applications, *Journal of Difference Equations and Applications,* Vol 18, Issue 6, 1001-1014.

[18] Bohner Martin, Guseinov Gusein Sh. (2010).The *h*-Laplace and *q*-Laplace transforms, *Journal of Mathematical Analysis and Applications,* Volume 365, Issue 1, 75-92.

[19] Purohit S.D., Kalla S.L. (2007). On *q*-Laplace Transforms of the q-Bessel functions, *Fractional Calculus and Applied Analysis,* Vol. 10, No. 2, 189-196.

[20] Kim Yong Sup, Lee Chang Hyun (2017). Evaluation of Some New Laplace Transforms for The Generalized Hypergeometric Function ${}_pF_p$, *Honam Mathematical Journal,* Vol. 39, No. 1, 73-82. MR3676675.

[21] Patra Baidyanath (2018). An Introduction to Integral Transforms (ISBN 978-1-138-58803-5), CRC Press, Boca Raton, FL 33487-2742. Zbl 06840501.

[22]. Srivastava H.M., Karlsson P.W. (1985). Multiple Gaussian Hypergeometric Series, Ellis Horwood Ltd., Chichester, U.K.